

# Equilibrium of stellar dynamical systems in the context of the Vlasov-Poisson model

Jérôme Perez

*Applied Mathematics Laboratory, Ecole Nationale Supérieure de Techniques  
Avancées, 32 Bd Victor, 75739 Paris Cedex 15*

---

## Abstract

This short review is devoted to the problem of the equilibrium of stellar dynamical systems in the context of the Vlasov-Poisson model. In a first part we will review some classical problems posed by the application of the Vlasov-Poisson model to the astrophysical systems like globular clusters or galaxies. In a second part we will recall some recent numerical results which may give us some quantitative hints about the equilibrium state associated to those systems.

*Key words:* Classical gravitation, Vlasov-Poisson system, Equilibrium

---

## 1 Introduction

Globular clusters and galaxies are concentration of stars whose physical characteristics are such that allow astrophysicists to model them as being in some equilibrium state for not too long time scales. The large number of stars and some properties of the gravitational interaction – which evidently play a key role for their dynamics – made the Vlasov-Poisson model a good candidate for their modeling. In this short review we present some results and problems of this field of research.

## 2 The Vlasov-Poisson system in the context of stellar dynamics

The Vlasov-Poisson model is a mean field approximation of the gravitational potential generated by a large assembly of point masses in the context of the

---

*Email address:* `jerome.perez@ensta.fr` (Jérôme Perez).

non dissipative kinetic theory. In order to apply this model to self-gravitating systems like globular clusters or galaxies, we have to pose three classical hypotheses : First of all, it suffices to consider only Newtonian gravity for the mean field interaction between stars <sup>1</sup>. Second, all stars are statistically equivalent, particularly they do have the same mass and they do not evolve. This is a drastic approximation but it could be relevant in a mean sense and during quite long evolution stages of these objects. Third, the considered systems must be non-dissipative on the time scales of interest. Nevertheless, it is well known that stellar dynamical systems could be dissipative mainly by two processes : On the one hand, such systems could contain gas which dissipates their total energy by dynamical friction. This feature must be considered for modeling the visible part of spiral galaxies. On the other hand it is well known that deviations from the mean-field forces due to the discreteness of the actual stellar distribution will alter the distribution of kinetic versus potential energies as computed from the Vlasov-Poisson equations. In fact, this is a problem of time scales. The dynamical time  $T_d$  of self-gravitating systems depends only on its mean density  $\tilde{\rho}$  via the relation  $T_d \approx 1/\sqrt{G\tilde{\rho}}$  where  $G$  is the Newton gravitational constant (see (6) for details). It is typically the time taken by a test star to cross the system. Another duration is under interest : the time  $T_r$  required for the mean velocity of the system to change by of order itself. This time has been estimated by Chandrasekhar for uniform systems to be proportional to the dynamical one via the relation  $T_r \approx N T_d / \ln N$  (see (6) p. 189 for details). It is typically the time for which the dissipative process like encounters plays a key role in the system's dynamics. Galaxies are composed of at least  $N = 10^9$  stars, therefore this time limitation is not relevant in this case. The situation is not so clear for globular clusters. The number of their components ranges from  $10^4$  to  $10^6$ . If initial stages (during a few hundred of dynamical times ...) could be non dissipative and therefore potentially described by Vlasov equation, during late stages – for billions years clusters – encounters dissipate energy and stellar dynamicists introduce Fokker-Planck formalism.

### 3 Equilibrium as steady state solutions of the Vlasov-Poisson system

Self-gravitating collisionless systems could be modeled by a phase space distribution function  $f(\mathbf{r}, \mathbf{p}, t)$  and a mean field potential  $\psi(\mathbf{r}, t)$ . Vectors  $\mathbf{r}$  and  $\mathbf{p}$  are the usual conjugated position and momentum which are elements of  $\mathbb{R}^d$

---

<sup>1</sup> The general theory of relativity could also be considered in the context of the cosmological principle. The system takes then into account the scale factor  $a(t)$  of the Universe. This formalism is often used in the context of formation of the large structures of the Universe.

for  $d$  dimensional systems. The functions  $f$  and  $\psi$  are coupled by the Vlasov-Poisson system, which is for  $d = 3$ :

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + \{ E, f \} = 0 \quad \text{where} \quad E = \frac{p^2}{2m} + m\psi \\ \psi(\mathbf{r}, t) = -Gm \int \frac{f(\mathbf{r}', \mathbf{p}', t)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' d\mathbf{p}' \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \Delta\psi = 4\pi G \overbrace{m \int f(\mathbf{r}, \mathbf{p}', t) d\mathbf{p}'}^{\rho(\mathbf{r}, t)} \\ \psi \text{ bounded at } \infty \end{array} \right.$$

The quantity  $E$  denotes the mean field energy of a test star and plays a central role in this system. The function  $\rho(\mathbf{r}, t)$  represent the mean mass density distribution of the system. Writing the Vlasov equation as above – i.e. using the Poisson brackets – makes trivial the well known result that every positive and normed function  $f_o$  of the mean field energy is a steady state solution of this system. A natural question, initially posed by S. Chandrasekhar in the middle of the last century, could then be : What are the properties of physical systems whose distribution function writes  $f_o = f_o(E)$  ? Although the answer of the inverse problem was well known by astrophysicists <sup>2</sup>, the natural direct one was solved only fifty years later (2), mainly by using in this context a difficult, but classical, mathematical theorem by Gidas, Ni and Nirenberg (1). The main ingredients of this result could be sketched as follow : If the distribution function depends only on the mean field energy, then the mean mass density depends on the position  $\mathbf{r}$  only through the mean field potential

$$f_o = f(E) \Rightarrow \rho_o(\mathbf{r}) = m \int f\left(\frac{p'^2}{2m} + m\psi_o\right) d\mathbf{p}' = \rho_o(\psi_o)$$

In this case, Poisson equation writes  $\Delta\psi_o = c \rho_o(\psi_o)$  where  $c$  is a positive constant. The physical context allows additional hypotheses : Newtonian gravity imposes that  $\psi_o(\mathbf{r})$  is a negative function bounded at infinity. The classical continuous limit let us consider that  $\rho_o(\mathbf{r})$  is a positive and continuous function. The GNN theorem (1) then allows us to claim that  $\psi_o = \psi_o(|\mathbf{r}|)$  and therefore the corresponding system has spherical symmetry in the spatial part of the phase space. Properties of the system in the velocity part of the phase space are more evident : The dispersion velocity tensor is clearly proportional to unity, therefore the system is said isotropic in velocity space. Spherical and isotropic steady state solutions of Vlasov-Poisson systems was intensively studied in the context of stellar dynamics as classical models for globular clusters

---

<sup>2</sup> As a matter of fact, a spherical self-gravitating system must be associated to a radial gravitationnal potential which produces naturally a radial density profile by Poisson equation

and galaxies. Stability of such systems was proved for monotonic distributions functions in the linear case (see (2) for a whole presentation of this topic) and also in some general non linear cases (5).

Considering other potential isolating integrals of motion in the gravitational field, we can define other classes of steady state solutions of Vlasov-Poisson systems : This classical property of Poisson brackets is sometimes called Jeans Theorem in stellar dynamics (see (6) for instance).

When the distribution function depend additionally on the mean angular momentum modulus  $f_o = f_o(E, L^2)$ , an extension of the GNN theorem shows that if the mass density is monotonic, then the system is always spherical. However, the tangential velocity dependence of  $L^2 = r^2 v_t^2$  makes the system anisotropic in the velocity space. The asymmetry of such systems is associated to an imbalance in the ratio of radial over tangential star orbits velocity distribution. Since the end of the 80's, stellar dynamicists have understood that such anisotropy could be at the origin of an interesting instability. As a matter of fact, if anisotropic spherical stellar systems are generally stable against radial perturbations , it could be proved that systems whose cannot be infinitesimally perturbed by radial disturbances are intrinsically unstable(4). This is the fine mechanism of Radial Orbit Instability : For pure radial orbit system, each star orbit extension is exactly the radius of the whole system. Thus, such a system *cannot* receive infinitesimal radial perturbation which affects by definition only an infinitesimal part of the system. On the contrary, any non radial perturbation associated to a given spatial direction could stretch or compress infinitesimally the spatial extension of an associated star orbit. This feature generates a tidal friction which makes an instability to grow and forms a triaxial system from an initial sphere. Radial Orbit Instability triggers when a sufficient amount of radial orbits is present in the system – a general criterion is given in (4) from distribution function susceptibility to receive radial perturbations. As indicated by numerical analysis (11), this feature could be at the origin of triaxiality in some self-gravitating systems like elliptical galaxies.

Less is known about steady state solutions characterized by distribution functions depending, in addition of  $E$ , on more complicated integrals. If some special models associated to  $f_o = f(E, L_z)$  are clearly triaxial (see (6) for details) , there is, up to now, no extensions of the famous GNN theorem which allows to claim anything in a general way.

## 4 Equilibrium of a stellar system

Taking into account the time problem limitation presented in section 2 and their generally quiet physical properties, one can modelize globular clusters and at least elliptical galaxies by steady state solutions of the Vlasov-Poisson system. The fundamental question that we have to answer is : What are the associated distribution functions ?

This problem was attacked by stellar dynamicists using three approaches : Thermodynamics, comparison with observational data and numerical simulation.

The thermodynamical approach, e.g. (13), is based on the classical assumption of statistical physics which associates the equilibrium state to the maximum Boltzmann entropy one. In the late sixties, (3) reconsider this problem and failed thoroughly : The classical result of these works is the isothermal sphere which distribution function is  $f_o = f(E) \propto \exp(\beta E)$ . This failure comes from the fact that in the  $3-D$  case, such entropy maximizer no exists. Nevertheless, interesting features come from such an analysis when the system is put in an unphysical box – see (10) for a review of this topic. Another problem is the fact that the Boltzmann entropy

$$S = - \int f \ln f \, d\mathbf{r} d\mathbf{p}$$

which is extremalized in this approach, is a conserved Casimir functional in the Vlasov-Poisson context <sup>3</sup> !

Observational approach consists in the integration in the models of observed properties of globular clusters or galaxies. A non exhaustive list of such a models is presented in classical text books like (6) or more recently (7). Most famous ones are the King model for globular clusters which is an arbitrarily truncated isothermal sphere, and the Navarro-Frenk-White(8) radial profile for dark matter halos associated to the galaxies. From a theoretical point of

---

<sup>3</sup> A Casimir functional is on the form  $\mathcal{C}[f] = \int C(f) d\mathbf{r} d\mathbf{p}$  where  $C$  is a smooth function. It is well known that such functionals are time conserved quantities if  $f$  is a solution of the Vlasov equation:

$$\begin{aligned} \frac{d\mathcal{C}[f]}{dt} &= \int \frac{dC(f)}{df} \frac{\partial f}{\partial t} d\mathbf{r} d\mathbf{p} = - \int \frac{dC(f)}{df} \left\{ \frac{\mathbf{p}}{m} \frac{\partial f}{\partial \mathbf{x}} - m \frac{\partial \psi}{\partial \mathbf{x}} \frac{\partial f}{\partial \mathbf{p}} \right\} d\mathbf{r} d\mathbf{p} \\ &= - \int \left\{ \frac{\mathbf{p}}{m} \frac{\partial C}{\partial \mathbf{x}} - m \frac{\partial \psi}{\partial \mathbf{x}} \frac{\partial C}{\partial \mathbf{p}} \right\} d\mathbf{r} d\mathbf{p} = 0 \end{aligned}$$

where the last equality follows by intergrating the first term over  $\mathbf{x}$  and the second term over  $\mathbf{p}$ , since  $f \rightarrow 0$  as  $|\mathbf{x}|, |\mathbf{p}| \rightarrow \infty$ .

view, all these models are poorly justified, except perhaps the Henon isochrone model (9) which is unfortunately generally unrealistic ...

The last way is numerical. Taking into account the recent ability given to stellar dynamicist by numerical tools to produce realistic formation experiences, important advances were made at the turning of the last century. These analysis are presented in the next section.

## 5 A numerical approach to equilibrium properties

The idea is to produce an equilibrium state from the gravitational collapse of an arbitrary physical set of point masses. This is not a recent idea and it was pioneered by Spitzer since the late sixties. The effective realization of such a study in a general way was the result of a serie of works displayed over more than twenty years ((11) and reference therein, (12) and reference therein). In those works , a general understanding of observed equilibrium properties of globular clusters and dark matter halos of galaxies follows from a detailed analysis of numerical gravitational collapses of a sufficiently large set of  $N$  point masses. Gravitational collapse from physical initial state produces generally a sphere. Flatness is possible by Radial Orbit Instability but it requires strong and inhomogeneous collapse <sup>4</sup>. The radial mass density profile of such spheres splits into two distinct classes : Homogeneous initial conditions collapses into a core halo-structure. It consists of a constant density region (the core) which extension can reach the half-mass radius of the system. This core is surrounded by a radial power law decreasing density region (the halo). The other class is composed by sufficiently inhomogeneous initial conditions whose collapse toward a monotonic radial power law decreasing density without notable central core. The origin of the collapsed core of such systems could be explained by the effect of the Antonov Instability discovered in the thermodynamical context (see (10) for a review). These two classes could be directly linked to the two classes of Newtonian self-gravitating systems which are globular clusters and galaxies. The classical hierarchical galaxy formation scenario is evidently linked to the inhomogeneous class. It produces collapsed core – which is a favorable process to form the observed and always mysterious super massive black holes – and potential flattening in violent cases. The smaller, quiet, more isolated and then homogeneous case, could be naturally interpreted as the generic globular cluster formation process. This could explain in the same operation, their generic spherical shape, their typical core-halo density profile, and finally their generic lack of intermediate mass black hole

---

<sup>4</sup> An important amount of solid rotation could also be invoked but this property is generally not observed in a sufficient way in stellar systems, like elliptical galaxies in particular.

in central region<sup>5</sup>. The case of spiral galaxies formation and evolution is more specific and the role of dissipative process cannot be neglected. Perhaps Vlasov equation is no more relevant for the description of such structures.

## 6 conclusion

Taking into account some physical constraints, the globular clusters and the galaxies could be suitably modeled by the Vlasov-Poisson model. In addition to this modelization, accurate numerical simulations allow to obtain a global understanding of important stages of their evolutions and of their main differences. However, a lot of fundamental works are always to be done before the study of very particular properties of the self-gravitating systems.

## Acknowledgements

The author thanks the referees for their very careful reading of the first version of the paper, and M. Kiessling for the communication of the fundamental reference (13).

## References

- [1] Gidas B, Ni W.-M. & Nirenberg L., *Symmetry of positive solutions of non linear elliptic equation in  $\mathbb{R}^3$* , Math. Anal. and Appl., **Part A**, 369, 1981
- [2] Perez J. & Aly J.-J., *Stability of spherical self-gravitating systems I - Analytic*, Monthly Not. Roy. Astron. Soc., **280**, 689, 1996
- [3] Lynden Bell D., *Statistical mechanics of violent relaxation in stellar systems*, Monthly Not. Roy. Astron. Soc., **136**, 101, 1967
- [4] Perez J. , Alimi J.-M., Aly J.-J. & Scholl H., *Stability of spherical self-gravitating systems II - Numerical*, Monthly Not. Roy. Astron. Soc., **280**, 700, 1996
- [5] Rein G., *Nonlinear Stability of Newtonian Galaxies and Stars from a Mathematical Perspective*, Ann. N. Y. Acad. Sci., **1045**, 103, 2005
- [6] Binney J. & Tremaine S., *Galactic Dynamics*, Princeton University Press, 1987

---

<sup>5</sup> The small observed amount of solid rotation could explain some observed flatness. Moreover, collapsed core of approximately 15% of the galactic globular clusters is due to dissipation effects related to the time problem noted in section 2 see (11) for details

- [7] Bertin G., *Dynamics of galaxies*, Cambridge University Press, 2000
- [8] Navarro J. F., Frenk C. S. & White S. D. M., *The Structure of Cold Dark Matter Halos*, The Astroph. J., **462**, 563, 1996
- [9] Henon M., *L'amas isochrone*, Ann. d'Astrop., **22**, 126, 1959
- [10] Perez J., *Gravity, Dimension, Equilibrium & Thermodynamics* Comptes Rendus Ac. Sc. Paris, **7**, 406, 2006
- [11] Roy F. & Perez J., *Dissipationless collapse of a set of  $N$  massive particles*, Monthly Not. Roy. Astron. Soc., **348**, 62, 2004
- [12] Boily C. M. & Athanassoula E., *On the equilibrium morphology of systems drawn from spherical collapse experiments*, Monthly Not. Roy. Astron. Soc., **369**, 608, 2006
- [13] Emden R., *Gaskugeln : Anwendungen der merchanischen Wärmethorie*, Leipzig, 1907